# Lecture 1: Aspects of probability theory.

## Gaussian Random Fields. (ref: Wandelt 2013)

- **Definition:**

$$p(x|\mu, C) = \frac{1}{\sqrt{|2\pi C|}} \exp\left(-\frac{1}{2}(x-\mu)^* C^{-1}(x-\mu)\right).$$

- **Integration by differentiation:**

$$\langle x \rangle = \mu \qquad \text{from solving } \partial_x \left(\ln p(x|\mu, C)\right) = 0.$$

$$\langle (x-\mu)^* (x-\mu) \rangle = C \qquad \text{from} \qquad \partial_x^2 \left(\ln p(x|\mu, C)\right) = -C^{-1}.$$

- **Generating a GRF:**
  - draw white noise $\xi$.
  - find the square root of $C$: $\sqrt{C}$ (any such matrix work).
  - $x = \sqrt{C}\,\xi + \mu$.

  **Notebook:** GRF_and_fNL

- **Higher-order moments of GRFs:**
  - any odd moment is 0 : $\langle x_1 x_2 x_3 \rangle = 0 \quad \langle x_1 x_2 x_3 x_4 x_5 \rangle = 0$.
  - even moments by Wick's theorem:

$$\langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2 \rangle\langle x_3 x_4 \rangle + \langle x_1 x_3 \rangle\langle x_2 x_4 \rangle + \langle x_1 x_4 \rangle\langle x_2 x_3 \rangle$$

$$= C_{12}\, C_{34} + C_{13}\, C_{24} + C_{14}\, C_{23}.$$

- **Marginals:**

$$\underset{\substack{\uparrow \\ \text{GRF}(\mu, C)}}{\lambda = \begin{pmatrix} \lambda_x \\ \lambda_y \end{pmatrix}} \qquad \mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \qquad C = \begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix}$$

then:

$$\langle \lambda_x \rangle = \mu_x. \qquad \langle (\lambda_x - \mu_x)^* (\lambda_x - \mu_x) \rangle = C_{xx}$$

$$\langle \lambda_y \rangle = \mu_y. \qquad \langle (\lambda_y - \mu_y)^* (\lambda_y - \mu_y) \rangle = C_{yy}.$$

The marginal means and covariances are just the corresponding parts of the joint mean and covariance.

- **Conditionals:**

$$\mu_{x|y} \equiv \langle \lambda_x | \lambda_y \rangle = \mu_x + C_{xy} C_{yy}^{-1} (\lambda_y - \mu_y).$$

$$C_{x|y} \equiv \langle (\lambda_x - \mu_x)^* (\lambda_x - \mu_x) | \lambda_y \rangle = C_{xx} - C_{xy} C_{yy}^{-1} C_{yx}.$$

$$\mu_{y|x} \equiv \langle \lambda_y | \lambda_x \rangle = \mu_y + C_{yx} C_{xx}^{-1} (\lambda_x - \mu_x).$$

$$C_{y|x} \equiv \langle (\lambda_y - \mu_y)^* (\lambda_y - \mu_y) | \lambda_x \rangle = C_{yy} - C_{yx} C_{xx}^{-1} C_{xy}.$$

demonstration : see e.g. Appendix A in Leclercq 2015.
(arXiv: 1512.04985)

- **Wiener Filtering: Denoising.** data model: $d = s + n$. $d, s, n$ are GRFs.

joint GRF: $z = \begin{pmatrix} d \\ s \end{pmatrix}$. assumptions: $\mu_d = \mu_s = \mu_n = 0$.
$C_{sn} = C_{ns}^* = 0$

$$\Rightarrow \begin{cases} C_{sd} = C_{ss} + C_{sn} = C_{ss}. \\ C_{dd} = C_{ss} + C_{nn}. \end{cases}$$

$\begin{pmatrix} C_{ss} \equiv S. \\ C_{nn} \equiv N. \end{pmatrix}$

WF equations:
$$\mu_{s|d} = C_{ss} (C_{ss} + C_{nn})^{-1} d. = S (S+N)^{-1} d.$$
$$C_{s|d} = C_{ss} - C_{ss} (C_{ss} + C_{nn})^{-1} C_{ss} = S - S (S+N)^{-1} S$$

other forms:
$$\mu_{s|d} = (S^{-1} + N^{-1})^{-1} N^{-1} d = S (\mathbb{1} + N^{-1} S)^{-1} N^{-1} d.$$

(check it)

$$C_{s|d} = (N^{-1} + S^{-1})^{-1} = S^{\frac{1}{2}} (\mathbb{1} + S^{\frac{1}{2}} N^{-1} S^{\frac{1}{2}}) S^{\frac{1}{2}}$$

**Notebook:** Denoising & Denoising (CMB).

- **Wiener Filtering: Deblending.** data model: $d = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} + a \begin{pmatrix} 0 \\ x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}.$

$$C_{x_1 d} = \begin{pmatrix} C_{x_1x_1} & C_{x_1x_1} & 0 \end{pmatrix} \qquad C_{x_2 d} = \begin{pmatrix} 0 & C_{x_2x_2} & C_{x_2x_2} \end{pmatrix}$$

$$C_{dd} = \begin{pmatrix} C_{x_1x_1} + C_{n_1n_1} & C_{x_1x_1} & 0 \\ C_{x_1x_1} & C_{x_1x_1} + C_{x_2x_2} + C_{n_2n_2} & C_{x_2x_2} \\ 0 & C_{x_2x_2} & C_{x_2x_2} + C_{n_3n_3} \end{pmatrix}.$$

**Notebook:** Deblending

$$\mu_{x_1|d} = C_{x_1 d} C_{dd}^{-1} d. \qquad C_{x_1|d} = C_{x_1x_1} - C_{x_1 d} C_{dd}^{-1} C_{d x_1}.$$
$$\mu_{x_2|d} = C_{x_2 d} C_{dd}^{-1} d \qquad C_{x_2|d} = C_{x_2x_2} - C_{x_2 d} C_{dd}^{-1} C_{d x_2}.$$