Chapter 1

Cosmological perturbations and structure formation

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"For the mind wants to discover by reasoning what exists in the infinity of space that lies out there, beyond the ramparts of this world – that region into which the intellect longs to peer and into which the free projection of the mind does actually extend its flight."
— Lucretius, *De Rerum Natura*

Abstract

This chapter provides an overview of the current paradigm of cosmic structure formation, as relevant for this thesis. It also reviews standard tools for large-scale structure analysis.

This chapter is organized as follows. In section 1.1, key equations of general relativistic Friedmann-Lemaître cosmological models are briefly reviewed, followed by a discussion of the statistical description of cosmological fields in section 1.2 and of the dynamics of gravitational instability in section 1.3. In section 1.4 and 1.5 we describe cosmological perturbation theory in Eulerian and Lagrangian descriptions. Finally, section 1.6 deals with various non-linear approximations to gravitational instability.

1.1 The homogeneous Universe

This section provides an overview of the standard picture of cosmology, describing the homogeneous evolution of the Universe. In particular, we reproduce some very standard equations around which perturbation theory will be implemented in the following. A demonstration can be found in any introduction to cosmology, see for example Peebles (1980); Kolb & Turner (1990); Liddle & Lyth (2000); Bernardeau *et al.* (2002); Lesgourgues (2004); Trodden & Carroll (2004); Langlois (2005, 2010).

Let a be the cosmic scale factor, normalized to unity at the present time: $a_0 = 1$. We denote by t the cosmic time, by τ the conformal time, defined by $\mathrm{d}t = a(\tau)\,\mathrm{d}\tau$, and by z the redshift, defined by $a = 1/(1+z)$. In the following, a dot denotes a differentiation with respect to t and a prime a differentiation with respect to τ. Friedmann's equations, describing the dynamics of the Universe, are derived from Einstein's equations of general relativity. In conformal time, they read:

$$\mathcal{H}^2 = \frac{8\pi\mathrm{G}}{3}a^2\rho - k, \tag{1.1}$$

$$\text{and} \quad \mathcal{H}' = -\frac{4\pi\mathrm{G}}{3}a^2(\rho + 3P), \tag{1.2}$$

where $\mathcal{H} \equiv a'/a = aH$ is the conformal expansion rate, $H \equiv \dot{a}/a$ is the Hubble parameter, ρ the total energy density and P the pressure. k is the reduced curvature parameter, taking one of the following values: -1 for an open universe, 0 for a flat universe, 1 for a closed universe. G denotes the gravitational constant, and we adopt units such that $\mathrm{c} = 1$.

As a direct consequence, Friedmann's equations immediately determine the evolution of the energy density, described as:

$$\rho' = -3\mathcal{H}(\rho + P). \tag{1.3}$$

Throughout this thesis, we will particularly focus on the eras of matter domination and dark-energy domination within the standard ΛCDM paradigm. Hence, we will consider that the content of the Universe is limited to two components: matter (mostly cold dark matter) and dark energy in the form of a cosmological constant Λ. We denote by ρ_m and $\rho_\Lambda \equiv \Lambda/8\pi\mathrm{G}$ their respective energy densities. Introducing their respective equations of state, $w_i \equiv P_i/\rho_i$, we have $w \approx 0$ for cold dark matter and $w = -1$ for the cosmological constant. For this cosmology, equation (1.2) reads

$$\mathcal{H}' = -\frac{4\pi\mathrm{G}}{3}a^2\rho_\mathrm{m} + \frac{8\pi\mathrm{G}}{3}a^2\rho_\Lambda. \tag{1.4}$$

It is convenient to introduce the dimensionless cosmological parameters as the ratio of density to critical density, $\rho_\mathrm{crit}(t) \equiv 3H^2(t)/8\pi\mathrm{G}$, which corresponds to the total energy density in a flat universe: $\Omega_\mathrm{m}(t) \equiv 8\pi\mathrm{G}\rho_\mathrm{m}(t)/3H^2(t)$ and $\Omega_\Lambda(t) \equiv 8\pi\mathrm{G}\rho_\Lambda/3H^2(t) = \Lambda/3H^2(t)$. Their expression in terms of conformal time is given by

$$\Omega_\mathrm{m}(\tau)\mathcal{H}^2(\tau) = \frac{8\pi\mathrm{G}}{3}\rho_\mathrm{m}(\tau)a^2(\tau), \tag{1.5}$$

$$\Omega_\Lambda(\tau)\mathcal{H}^2(\tau) = \frac{8\pi\mathrm{G}}{3}\rho_\Lambda a^2(\tau) \equiv \frac{\Lambda}{3}a^2(\tau). \tag{1.6}$$

Note that $\Omega_\mathrm{m}(\tau)$ and $\Omega_\Lambda(\tau)$ are time-dependent. Inserting these two expressions in equation (1.4) yields the following form of the second Friedmann equation,

$$\mathcal{H}'(\tau) = \left(-\frac{\Omega_\mathrm{m}(\tau)}{2} + \Omega_\Lambda(\tau)\right)\mathcal{H}^2(\tau), \tag{1.7}$$

and the first one reads

$$\mathcal{H}^2 = \frac{8\pi\mathrm{G}}{3}a^2\rho_\mathrm{m} + \frac{8\pi\mathrm{G}}{3}a^2\rho_\Lambda - k = \Omega_\mathrm{m}\mathcal{H}^2 + \Omega_\Lambda\mathcal{H}^2 - k, \tag{1.8}$$

which yields

$$k = (\Omega_\mathrm{tot}(\tau) - 1)\mathcal{H}^2(\tau), \tag{1.9}$$

where $\Omega_\mathrm{tot}(\tau) \equiv \Omega_\mathrm{m}(\tau) + \Omega_\Lambda(\tau)$. In the following, we will note $\Omega_\mathrm{m}^{(0)} = \Omega_\mathrm{m}(a=1)$ and $\Omega_\Lambda^{(0)} = \Omega_\Lambda(a=1)$.

1.2 Statistical description of cosmological fields

In this section, we consider some cosmic scalar field $\lambda(\mathbf{x})$ whose statistical properties are to be described. It denotes either the cosmological density contrast, $\delta(\mathbf{x})$, the gravitational potential, $\Phi(\mathbf{x})$ (see section 1.3), or any other field of interest derived from vectorial fields (e.g. the velocity divergence field), polarization fields, etc.

As discussed in the introduction, values of $\lambda(\mathbf{x})$ have to be treated as stochastic variables. For an arbitrary number n of spatial positions $\mathbf{x}_i$, one can define the *joint multivariate probability distribution function* to have $\lambda(\mathbf{x}_1)$ between λ_1 and $\lambda_1 + \mathrm{d}\lambda_1$, $\lambda(\mathbf{x}_2)$ between λ_2 and $\lambda_2 + \mathrm{d}\lambda_2$, etc. This pdf is written

$$\mathcal{P}(\lambda_1, \lambda_2, ..., \lambda_n)\,\mathrm{d}\lambda_1\mathrm{d}\lambda_2 \ldots \mathrm{d}\lambda_n. \tag{1.10}$$

1.2.1 Average and ergodicity

Average. The word "average" (and in the following, the corresponding $\langle\rangle$ symbols) may have two different meanings. First, one can average by taking many realizations drawn from the distribution, all of them produced in the same way (e.g. by N-body simulations). This is the *ensemble average*, defined to be for any quantity $X(\lambda_1, \lambda_2, ..., \lambda_n)$:

$$\langle X \rangle \equiv \int X(\lambda_1, \lambda_2, ..., \lambda_n)\mathcal{P}(\lambda_1, \lambda_2, ..., \lambda_n)\,\mathrm{d}\lambda_1\mathrm{d}\lambda_2 \ldots \mathrm{d}\lambda_n, \tag{1.11}$$

where $\mathcal{P}(\lambda_1, \lambda_2, ..., \lambda_n)$ is the joint multivariate pdf.

One can also average by considering the quantity of interest at different locations within the same realization of the distribution. This is the *sample average*. For some volume V in the Universe, the sample average over V of a quantity X is defined to be:

$$\bar{X} \equiv \frac{1}{V}\int_V X(\mathbf{x})\,\mathrm{d}^3\mathbf{x}. \tag{1.12}$$

Ergodicity. If the ensemble average of any quantity coincides with the sample average of the same quantity, the system is said to be *ergodic*. In cosmology, the hypothesis of ergodicity is often adopted, at least if the considered catalogue is large enough. For instance, if ergodicity holds, the mean density of the Universe is given by

$$\langle\rho(\mathbf{x})\rangle = \bar{\rho} \equiv \frac{1}{V}\int_V \rho(\mathbf{x})\,\mathrm{d}^3\mathbf{x}, \tag{1.13}$$

in the limit where $V \rightarrow \infty$. The term of ergodicity historically refers to time processes, not to spatial ones. If the above property is fulfilled in cosmology, one says that the system is a *fair sample* of the Universe.[1]

1.2.2 Statistical homogeneity and isotropy

A random field is said to be *statistically homogeneous* if all joint multivariate pdfs $\mathcal{P}(\lambda(\mathbf{x}_1), \lambda(\mathbf{x}_2), ..., \lambda(\mathbf{x}_n))$ are invariant under translations of the coordinates $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n$ in space. Thus probabilities depend only on relative positions, but not on locations. Note that statistical homogeneity is a weaker assumption than homogeneity, which would mean that $\lambda(\mathbf{x})$ takes the same value everywhere in space.

Similarly, a random field is said to be *statistically isotropic* if all $\mathcal{P}(\lambda(\mathbf{x}_1), \lambda(\mathbf{x}_2), ..., \lambda(\mathbf{x}_n))$ are invariant under spatial rotations.

From now on, cosmic fields will be considered statistically homogeneous and isotropic, as a consequence of the cosmological principle that underlies most inflationary calculations (see Guth, 1981; Linde, 1982; Albrecht & Steinhardt, 1982; Linde, 1995), and of standard gravitational evolution (e.g. Peebles, 1980). Of course, the validity of this assumption has to be checked against observational data. It is also important to note that a lot of the information from galaxy surveys comes from effects that distort the observed signal away from this ideal. In particular, observational effects such as the Alcock-Paczynski effect (Alcock & Paczynski, 1979) and redshift-space distortions (Kaiser, 1987) in galaxy surveys introduce significant deviations from statistical homogeneity and isotropy.

[1] In the case of the LSS, care should be taken with deep surveys. Indeed, as data lie on the surface of the relativistic lightcone, we cannot have access to a fair sample of the Universe at the present time. Rigorously, ergodicity is not verified.

1.2.3 Gaussian and log-normal random fields in cosmostatistics

This section draws from subsection 2.2 of Leclercq, Pisani & Wandelt (2014).

Gaussian random fields are ubiquitous in cosmostatistics (see Lahav & Suto, 2004; Wandelt, 2013; Leclercq, Pisani & Wandelt, 2014, for reviews). Indeed, as mentioned in the introduction, inflationary models predict the initial density perturbations to arise from a large number of independent quantum fluctuations, and therefore to be very nearly Gaussian-distributed. Even in models which are said to produce "large" non-Gaussianities, deviations from Gaussianity are strongly constrained by observational tests (see Planck Collaboration, 2014b, 2015, for the latest results). Grfs are essential for the analysis of the cosmic microwave background, but the large scale distribution of galaxies can also be approximately modeled as a grf, at least on very large scales, where gravitational evolution is still well-described by linear perturbation theory (see sections 1.4 and 1.5). The log-normal distribution is convenient for modeling the statistical behavior of evolved density fields, partially accounting for non-linear gravitational effects at the level of the one-point distribution.

In the following, we summarize some results about finite-dimensional Gaussian and log-normal random fields. Without loss of generality, infinite-dimensional fields can be discretized. If the field is sufficiently regular and the discretization scale is small enough, no information will be lost. In practice, throughout this thesis, any field that we want to describe is already discretized on a grid of particles or voxels. Let us denote the values of the considered cosmic scalar field $\lambda(\mathbf{x})$ at comoving positions $\mathbf{x}_i$ as $\lambda_i \equiv \lambda(\mathbf{x}_i)$ for i from 1 to any arbitrary integer n.

1.2.3.1 Gaussian random fields

The n-dimensional vector $\lambda = (\lambda_i)_{1\leq i\leq n}$ is a Gaussian random field (we will often say "is Gaussian" in the following) with mean $\mu \equiv (\mu_i)_{1\leq i\leq n}$ and covariance matrix $C \equiv (C_{ij})_{1\leq i\leq n, 1\leq j\leq n}$ if its joint multivariate pdf is a multivariate Gaussian:[2]

$$\mathcal{P}(\lambda|\mu, C) = \frac{1}{\sqrt{|2\pi C|}} \exp\left(-\frac{1}{2}(\lambda-\mu)^* C^{-1}(\lambda-\mu)\right) = \frac{1}{\sqrt{|2\pi C|}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\lambda_i-\mu_i)C^{-1}_{ij}(\lambda_j-\mu_j)\right). \tag{1.14}$$

where z^* denotes the conjugate transpose of z, vertical bars indicate the determinant of the surrounded matrix and C is assumed to be a positive-definite Hermitian matrix (and therefore invertible). In practical cases, μ is often taken to be zero. As can be seen from this definition, a grf is completely specified by its mean μ and its covariance matrix C.

It is interesting to note that for the density contrast $\delta(\mathbf{x})$, the Gaussian assumption has to break down at later epochs of structure formation since it predicts density amplitudes to be symmetrically distributed among positive and negative values, but weak and strong energy conditions require $\delta(\mathbf{x}) \geq -1$. Even in the initial conditions, Gaussianity can not be exact due to the existence of this lower bound. The Gaussian assumption is therefore strictly speaking only valid in the limit of infinitesimally small density fluctuations, $|\delta(\mathbf{x})| \ll 1$.

1.2.3.2 Moments of Gaussian random fields, Wick's theorem

From equation (1.14) it is easy to check that the mean $\langle\lambda\rangle$ is really μ and the covariance matrix is really $\langle(\lambda-\mu)^*(\lambda-\mu)\rangle = C$, just by evaluating the Gaussian integrals:

$$\langle\lambda_i\rangle = \int \lambda_i \, \mathcal{P}(\lambda|\mu, C)\, \mathrm{d}\lambda_i = \mu_i, \tag{1.15}$$

$$\langle(\lambda_i-\mu_i)^*(\lambda_j-\mu_j)\rangle = \int (\lambda_i-\mu_i)^*(\lambda_j-\mu_j)\, \mathcal{P}(\lambda|\mu, C)\, \mathrm{d}\lambda_i \mathrm{d}\lambda_j = C_{ij}. \tag{1.16}$$

We now want to compute higher-order moments of a grf. Let us focus on central moments, or equivalently, let us assume in the following that the mean is $\mu = 0$. Here we omit the star denoting conjugate transpose for simplicity. Any odd moments, e.g. the third $\langle\lambda_i\lambda_j\lambda_k\rangle$, the fifth $\langle\lambda_i\lambda_j\lambda_k\lambda_l\lambda_m\rangle$, etc. are found to be zero

[2] Here we use the common terminology in physics and refer to this pdf as a "multivariate Gaussian". It is called a "multivariate normal" distribution in statistics. Note that it is possible to generalize this definition to the case where C is only a positive semi-definite Hermitian matrix, using the notion of characteristic function (see appendix A).

by symmetry of the Gaussian pdf. The higher-order even ones (e.g. the fourth, the sixth, etc.) can be evaluated through the application of Wick's theorem, an elegant method of reducing high-order statistics to a combinatorics problem.

Wick's theorem states that high-order even moments of a grf are computed by connecting up all possible pairs of the field (Wick contractions) and writing down the covariance matrix for each pair using equation (1.16). For instance,

$$\begin{aligned} \langle \lambda_i \lambda_j \lambda_k \lambda_l \rangle &= \langle \lambda_i \lambda_j \rangle \langle \lambda_k \lambda_l \rangle + \langle \lambda_i \lambda_k \rangle \langle \lambda_j \lambda_l \rangle + \langle \lambda_i \lambda_l \rangle \langle \lambda_j \lambda_k \rangle \\ &= C_{ij} C_{kl} + C_{ik} C_{jl} + C_{il} C_{jk}. \end{aligned} \tag{1.17}$$

The number of terms generated in this fashion for the n-th order moment is $\prod_{i=1}^{n/2} (2i-1)$.

1.2.3.3 Marginals and conditionals of Gaussian random fields

Let us the split the grf up into two parts $x = [\![1, m]\!]$ and $y = [\![m+1, n]\!]$ $(m < n)$, so that

$$\lambda = \begin{pmatrix} \lambda_x \\ \lambda_y \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{pmatrix}. \tag{1.18}$$

$C_{xy} = (C_{yx})^*$ since C is Hermitian.

Easy computation of marginal and conditional pdfs is a very convenient property of grfs. First of all, marginal and conditional densities of grfs are multivariate Gaussians. Therefore, all we need to calculate are their means and covariances. For the marginal pdfs, the results are

$$\langle \lambda_x \rangle = \mu_x, \tag{1.19}$$

$$\langle (\lambda_x - \mu_x)^* (\lambda_x - \mu_x) \rangle = C_{xx}, \tag{1.20}$$

$$\langle \lambda_y \rangle = \mu_y, \tag{1.21}$$

$$\langle (\lambda_y - \mu_y)^* (\lambda_y - \mu_y) \rangle = C_{yy}. \tag{1.22}$$

These expressions simply mean that the marginal means and marginal covariances are just the corresponding parts of the joint mean and covariance, as defined by equation (1.18).

Less trivially, here are the parameters of the conditional densities:

$$\mu_{x|y} \equiv \langle \lambda_x | \lambda_y \rangle = \mu_x + C_{xy} C_{yy}^{-1} (\lambda_y - \mu_y), \tag{1.23}$$

$$C_{x|y} \equiv \langle (\lambda_x - \mu_x)^* (\lambda_x - \mu_x) | \lambda_y \rangle = C_{xx} - C_{xy} C_{yy}^{-1} C_{yx}, \tag{1.24}$$

$$\mu_{y|x} \equiv \langle \lambda_y | \lambda_x \rangle = \mu_y + C_{yx} C_{xx}^{-1} (x - \mu_x), \tag{1.25}$$

$$C_{y|x} \equiv \langle (\lambda_y - \mu_y)^* (\lambda_y - \mu_y) | \lambda_x \rangle = C_{yy} - C_{yx} C_{xx}^{-1} C_{xy}. \tag{1.26}$$

A demonstration of these formulae can be found in appendix A. From these expressions, it is easy to see that for grfs, lack of covariance ($C_{xy} = 0$) implies independence, i.e. $\mathcal{P}(x, y) = \mathcal{P}(x)\mathcal{P}(y)$. This is most certainly not the case for general random fields. Similarly, if $\lambda_1, ..., \lambda_n$ are jointly Gaussian, then each λ_i is Gaussian-distributed, but not conversely.

A particular case is the optimal de-noising of a data set d, modeled as the sum of some signal s and a stochastic noise contribution n: $d = s + n$. We model all three fields d, s, n as grfs. Assuming a vanishing signal mean, an unbiased measurement (i.e. $\mu_d = \mu_s = \mu_n = 0$), and lack of covariance between signal and noise (i.e. $C_{sn} = C_{ns} = 0$, which implies $C_{sd} = C_{ss}$ and $C_{dd} = C_{ss} + C_{nn}$), equations (1.23) and (1.24) yield the famous Wiener filter equations:

$$\mu_{s|d} = C_{ss} \left(C_{ss} + C_{nn}\right)^{-1} d, \tag{1.27}$$

$$C_{s|d} = C_{ss} - C_{ss} \left(C_{ss} + C_{nn}\right)^{-1} C_{ss} = \left(C_{nn}^{-1} + C_{ss}^{-1}\right)^{-1}. \tag{1.28}$$

1.2.3.4 Log-normal random fields

In the case where $\delta(\mathbf{x})$ is the density contrast, gravitational evolution will yield very high positive density contrast amplitudes. In order to prevent negative mass ($\delta(\mathbf{x}) < -1$) while preserving $\langle \delta(\mathbf{x}) \rangle = 0$, the resultant pdf must be strongly skewed (e.g. Peacock, 1999). In the absence of an exact pdf for the density field in non-linear regimes, solution to dynamical equations, one can describe the statistical properties of the evolved matter distribution by phenomenological probability distributions. A common choice is the log-normal distribution, which approximates well the one-point behavior observed in galaxy observations and N-body simulations (e.g. Hubble, 1934; Peebles, 1980; Coles & Jones, 1991; Gaztañaga & Yokoyama, 1993; Colombi, 1994; Kayo, Taruya & Suto, 2001; Neyrinck, Szapudi & Szalay, 2009). This is the model adopted as a prior for the density field in the HADES algorithm (Jasche & Kitaura, 2010; Jasche & Wandelt, 2012, see also table 4.2). The assumption is that the log-density, $\ln(1+\delta)$, instead of the density contrast δ, obeys Gaussian statistics.

If λ is a n-dimensional vector having multivariate Gaussian distribution with mean μ and covariance matrix C, then ξ, defined by its components $\xi_i = \exp(\lambda_i)$, has a multivariate log-normal distribution given by

$$\mathcal{P}(\xi|\mu, C) = \frac{1}{\sqrt{|2\pi C|}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (\ln(\xi_i) - \mu_i)^* \, C^{-1}_{ij} \, (\ln(\xi_j) - \mu_j)\right) \prod_k \frac{1}{\xi_k}. \tag{1.29}$$

The mean of ξ is ν defined by

$$\nu_i \equiv \langle \xi_i \rangle = \exp\left(\mu_i + \frac{1}{2}C_{ii}\right), \tag{1.30}$$

and its covariance matrix is D defined by

$$D_{ij} \equiv \langle (\xi_i - \mu_i)^*(\xi_j - \mu_j)\rangle = \exp\left(\mu_i + \mu_j + \frac{1}{2}(C_{ii} + C_{jj})\right)(\exp(C_{ij}) - 1). \tag{1.31}$$

In cosmology, we assume that $\lambda = \ln(1+\delta)$ is a grf with mean μ and covariance matrix C. Then $\delta = \exp(\lambda) - 1$ follows a log-normal distribution, given by

$$\mathcal{P}(\delta|\mu, C) = \frac{1}{\sqrt{|2\pi C|}} \exp\left(-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} (\ln(1+\delta_i) - \mu_i)^* \, C^{-1}_{ij} \, (\ln(1+\delta_j) - \mu_j)\right) \prod_k \frac{1}{1+\delta_k}. \tag{1.32}$$

To ensure that $\langle \delta \rangle$ vanishes everywhere, i.e. that

$$\nu_i = \langle 1 + \delta_i \rangle = \exp\left(\mu_i + \frac{1}{2}C_{ii}\right) = 1, \tag{1.33}$$

one has to impose the following choice for μ:

$$\mu_i = -\frac{1}{2}C_{ii} = -\frac{1}{2}C_{00} = \mu_0. \tag{1.34}$$

We have used that $C_{ii} = C_{00}$, since the correlation function depends only on distance (assuming statistical homogeneity and isotropy). Hence, the mean for the lognormal distribution is the same throughout the entire field.

For further discussion on the log-normal behavior of density fields, see chapters 2 and 6.

1.2.4 Correlation functions and power spectra

1.2.4.1 Two-point correlation function and power spectrum

Definitions. The two-point correlation function is defined in configuration space as the joint ensemble average of the field at two different locations:

$$\xi(r) = \langle \lambda^*(\mathbf{x})\lambda(\mathbf{x}+\mathbf{r})\rangle. \tag{1.35}$$

It depends only on the norm of $\mathbf{r}$ if statistical isotropy and homogeneity hold.

The scalar field $\lambda(\mathbf{x})$ is usually written in terms of its Fourier components,

$$\lambda(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int \lambda(\mathbf{k}) \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{x}) \, \mathrm{d}^3\mathbf{k}, \tag{1.36}$$

or, equivalently,

$$\lambda(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int \lambda(\mathbf{x}) \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x}) \, \mathrm{d}^3\mathbf{x}. \tag{1.37}$$

The quantities $\lambda(\mathbf{k})$ are complex random variables. If $\lambda(\mathbf{x})$ is real, one has $\lambda(-\mathbf{k}) = \lambda^*(\mathbf{k})$ which means that half of the Fourier space contains redundant information.

The computation of the two-point correlator for $\lambda(\mathbf{k})$ in Fourier space gives:

$$\begin{aligned}
\langle\lambda^*(\mathbf{k})\lambda(\mathbf{k}')\rangle &= \frac{1}{(2\pi)^{3/2}}\frac{1}{(2\pi)^{3/2}}\iint \langle\lambda^*(\mathbf{x})\lambda(\mathbf{x}+\mathbf{r})\rangle \exp(\mathrm{i}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x} - \mathrm{i}\mathbf{k}'\cdot\mathbf{r})\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{r} && (1.38)\\
&= \frac{1}{(2\pi)^3}\iint \xi(r)\exp(\mathrm{i}(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x} - \mathrm{i}\mathbf{k}'\cdot\mathbf{r})\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{r} && (1.39)\\
&= \frac{1}{(2\pi)^3}\,\delta_\mathrm{D}(\mathbf{k}-\mathbf{k}')\int \xi(r)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}^3\mathbf{r} && (1.40)\\
&\equiv \frac{1}{(2\pi)^{3/2}}\,\delta_\mathrm{D}(\mathbf{k}-\mathbf{k}')\,P(k), && (1.41)
\end{aligned}$$

where δ_D is a Dirac delta distribution and

$$P(k) \equiv \frac{1}{(2\pi)^{3/2}}\int \xi(r)\exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}^3\mathbf{r} \tag{1.42}$$

is defined to be the *power spectrum* of the field $\lambda(\mathbf{x})$ (this relation is known as the Wiener-Khinchin theorem). Because of statistical homogeneity and isotropy, it depends only on the norm of $\mathbf{k}$. The inverse relation between the two-point correlation function, $\xi(r)$, and the power spectrum, $P(k)$, reads

$$\xi(r) = \frac{1}{(2\pi)^{3/2}}\int P(k)\exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r})\,\mathrm{d}^3\mathbf{k}. \tag{1.43}$$

In spherical coordinates, using

$$\int_\Omega \exp\left(-\mathrm{i}kr\cos\theta\right)\mathrm{d}\Omega = \int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi}\exp\left(-\mathrm{i}kr\cos\theta\right)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi = 4\pi\frac{\sin(kr)}{kr}, \tag{1.44}$$

we obtain the one-dimensional relations between $\xi(r)$ and $P(k)$,

$$\begin{aligned}
P(k) &= \frac{2}{\sqrt{\pi}}\int_0^\infty \xi(r)\,j_0(kr)\,r^2\,\mathrm{d}r, && (1.45)\\
\xi(r) &= \frac{2}{\sqrt{\pi}}\int_0^\infty P(k)\,j_0(kr)\,k^2\,\mathrm{d}k, && (1.46)
\end{aligned}$$

where j_0 is the zero-th order spherical Bessel function,

$$j_0(x) \equiv \frac{\sin(x)}{x}. \tag{1.47}$$

Two-point probability function and two-point correlation function. The following physical interpretation of the two-point correlation function establishes a link between the ensemble average and the sample average. Indeed, correlation functions are directly related to multivariate probability functions (in fact, they are sometimes defined from them). Here we exemplify this fact for the density contrast at position $\mathbf{x}$, $\delta(\mathbf{x}) \equiv \rho(\mathbf{x})/\bar{\rho} - 1$.

Let us consider two infinitesimal volumes $\mathrm{d}V_1$ and $\mathrm{d}V_2$ inside the volume V. Let n_1 and n_2 be the particle densities at locations $\mathbf{x}_1$ and $\mathbf{x}_2$ and $n \equiv N/V$ the average numerical density. Then the density contrasts are $\delta(\mathbf{x}_1) = n_1/(n\,\mathrm{d}V_1) - 1$ and $\delta(\mathbf{x}_2) = n_2/(n\,\mathrm{d}V_2) - 1$ and the two-point correlation function reads

$$\xi(x_{12}) = \langle\delta(\mathbf{x}_1)\delta(\mathbf{x}_2)\rangle = \frac{\mathrm{d}N_{12}}{n^2\,\mathrm{d}V_1\,\mathrm{d}V_2} - 1, \tag{1.48}$$

where $x_{12} \equiv |\mathbf{x}_2 - \mathbf{x}_1|$ and $\mathrm{d}N_{12} = \langle n_1 n_2\rangle$ is the average number of *pairs* in the volumes $\mathrm{d}V_1$ and $\mathrm{d}V_2$ (i.e. the product of the number of particles in one volume times the number in the other volume). One can then rewrite

$$\mathrm{d}N_{12} = \langle n_1 n_2\rangle = n^2\left(1 + \xi(x_{12})\right)\mathrm{d}V_1\,\mathrm{d}V_2. \tag{1.49}$$

The physical interpretation of the two-point correlation function is that it measures the excess over uniform probability that two particles at volume elements $\mathrm{d}V_1$ and $\mathrm{d}V_2$ are separated by a distance x_{12}. If particle positions are drawn from uniform distributions (i.e. if there is no clustering), then $\mathrm{d}N_{12}$ is independent of the separation. In this case, the average number of pairs is the product of the average number of particles in the two volumes, $\langle n_1 n_2 \rangle = \langle n_1 \rangle \langle n_2 \rangle = n^2 \, \mathrm{d}V_1 \, \mathrm{d}V_2$ and the correlation ξ vanishes. Particles are said to be uncorrelated. Conversely, if ξ is non-zero, particle distributions are said to be correlated (if $\xi > 0$) or anti-correlated (if $\xi < 0$).

It is sometimes easier to derive the correlation function as the average density of particles at a distance r from another particle, i.e. by choosing the volume element $\mathrm{d}V_1$ such as $n \, \mathrm{d}V_1 = 1$. Then the number of pairs is given by the number of particles in volume $\mathrm{d}V_2$:

$$\mathrm{d}N_2 = n \, (1 + \xi(r)) \, \mathrm{d}V_2. \tag{1.50}$$

Hence, one can evaluate the correlation function as follows:

$$\xi(r) = \frac{\mathrm{d}N(r)}{n \, \mathrm{d}V} - 1 = \frac{\langle n(r) \rangle}{n} - 1, \tag{1.51}$$

i.e. as the average number of particles at distance r from any given particle, divided by the expected number of particles at the same distance in a uniform distribution, minus one. As $\mathrm{d}N_2$ is linked to the conditional probability that there is a particle in $\mathrm{d}V_2$ given that there is one in $\mathrm{d}V_1$, the previous expression is sometimes referred to as the *conditional density contrast*.

Two-point correlation function and power spectrum of Gaussian fields. If $\lambda(\mathbf{x})$ is a real grf of mean 0 and covariance matrix C, then equation (1.16) means that its two-point correlation function in configuration space is directly given by the covariance matrix: $\langle \lambda(\mathbf{x}_i) \lambda(\mathbf{x}_j) \rangle = C_{ij}$.

Additionally, if $\lambda(\mathbf{x})$ is also statistically homogeneous, equation (1.41) implies that $\lambda(\mathbf{k})$ has independent Fourier modes and that its covariance matrix in Fourier space is diagonal and contains the power spectrum coefficients $P(k)/(2\pi)^{3/2}$. Finally, according to Wick's theorem (section 1.2.3.2), one can write for any integer p:

$$\langle \lambda(\mathbf{k}_1) \ldots \lambda(\mathbf{k}_{2p+1}) \rangle = 0, \tag{1.52}$$

$$\begin{aligned} \langle \lambda(\mathbf{k}_1) \ldots \lambda(\mathbf{k}_{2p}) \rangle &= \sum_{\text{all pair associations}} \prod_{p \text{ pairs } (i,j)} \langle \lambda(\mathbf{k}_i) \lambda(\mathbf{k}_j) \rangle \\ &= \sum_{\text{all pair associations}} \prod_{p \text{ pairs } (i,j)} \delta_\mathrm{D}(\mathbf{k}_i - \mathbf{k}_j) \frac{P(k_i)}{(2\pi)^{3/2}}. \end{aligned} \tag{1.53}$$

Hence, for grfs, all statistical properties are included in two-point correlations. More specifically, all statistical properties of random variables $\lambda(\mathbf{k})$ are conclusively determined by the shape of the power spectrum $P(k)$.

1.2.4.2 Higher-order correlation functions

Higher-order correlation functions in configuration space. It is possible to define higher-order correlation functions, as the *connected part* (denoted by a subscript c) of the joint ensemble average of the field $\lambda(\mathbf{x})$ in an arbitrary number of locations. This can be formally written as

$$\begin{aligned} \xi_n(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) &= \langle \lambda(\mathbf{x}_1) \lambda(\mathbf{x}_2) \ldots \lambda(\mathbf{x}_n) \rangle_\mathrm{c} \\ &\equiv \langle \lambda(\mathbf{x}_1) \lambda(\mathbf{x}_2) \ldots \lambda(\mathbf{x}_n) \rangle - \sum_{\mathcal{S} \in \mathcal{P}(\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\})} \prod_{s_i \in \mathcal{S}} \xi_{\# s_i}(\mathbf{x}_{s_i(1)}, \mathbf{x}_{s_i(2)}, \ldots, \mathbf{x}_{s_i(\# s_i)}), \end{aligned} \tag{1.54}$$

where the sum is made over the proper partitions (any partition except the set itself) of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and s_i is a subset of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ contained in partition $\mathcal{S}$. When the average of $\lambda(\mathbf{x})$ is zero, only partitions that contain no singlets contribute. The decomposition in connected and non-connected parts of the joint ensemble average of the field can be easily visualized in a diagrammatic way (see e.g. Bernardeau *et al.*, 2002).

For grfs, as a consequence of Wick's theorem (section 1.2.3.2), all connected correlations functions are zero except ξ_2.

Higher-order correlators in Fourier space. This definition in configuration space can be extended to Fourier space. By statistical isotropy of the field, the n-th Fourier-space correlator does not depend on the direction of the $\mathbf{k}$-vectors. By statistical homogeneity, the $\mathbf{k}$-vectors have to sum up to zero. We can thus define $P_n(\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n)$ by

$$\langle \lambda(\mathbf{k}_1)\lambda(\mathbf{k}_2) \ldots \lambda(\mathbf{k}_n)\rangle_\mathrm{c} \equiv \delta_\mathrm{D}(\mathbf{k}_1 + \mathbf{k}_2 + \ldots + \mathbf{k}_n)\, P_n(\mathbf{k}_1, \mathbf{k}_2, ..., \mathbf{k}_n). \tag{1.55}$$

The Dirac delta distribution δ_D ensures that $\mathbf{k}$-vector configurations form closed polygons: $\sum_i \mathbf{k}_i = \mathbf{0}$.

Let us now examine the lowest-order connected moments.

Bispectrum. After the power spectrum, the second statistic of interest is the bispectrum B, for $n = 3$, defined by

$$\langle \lambda(\mathbf{k}_1)\lambda(\mathbf{k}_2)\lambda(\mathbf{k}_3)\rangle = \langle \lambda(\mathbf{k}_1)\lambda(\mathbf{k}_2)\lambda(\mathbf{k}_3)\rangle_\mathrm{c} \equiv \delta_\mathrm{D}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\, B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3). \tag{1.56}$$

Reduced bispectrum. It is convenient to define the reduced bispectrum $Q(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ as

$$Q(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) \equiv \frac{B(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{P(k_1)P(k_2) + P(k_1)P(k_3) + P(k_2)P(k_3)}, \tag{1.57}$$

which takes away most of the dependence on scale and cosmology. The reduced bispectrum is useful for comparing different models, because its weak dependence on cosmology is known to break degeneracies between cosmological parameters and to isolate the effects of gravity (see Gil-Marín *et al.*, 2011, for an example).

Trispectrum. The trispectrum is the following correlator, for $n = 4$. It is defined as

$$\langle \lambda(\mathbf{k}_1)\lambda(\mathbf{k}_2)\lambda(\mathbf{k}_3)\lambda(\mathbf{k}_4)\rangle_\mathrm{c} \equiv \delta_\mathrm{D}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4)\, T(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4). \tag{1.58}$$

1.3 Dynamics of gravitational instability

The standard picture for the formation of the LSS as seen in galaxy surveys is the gravitational amplification of primordial density fluctuations. The dynamics of this process is mostly governed by gravitational interactions of collisionless (or at least, weakly-interacting) dark matter particles in an expanding universe.

For scales much smaller than the Hubble radius, relativistic effects (such as the curvature of the Universe or the apparent distance-redshift relation) are believed to be subdominant or negligible (e.g. Kolb & Turner, 1990, and references therein) and as we will show, the expansion of the background can be factored out by a redefinition of variables. Although the microscopic nature of dark matter particles remains unknown, candidates have to pass several tests in order to be viable (Taoso, Bertone & Masiero, 2008). In particular, particles which are relativistic at the time of structure formation lead to a large damping of small-scale fluctuations (Silk, 1968; Bond & Szalay, 1983), incompatible with observed structures. The standard theory thus requires dark matter particles to be cold during structure formation, i.e. non-relativistic well before the matter-dominated era (Peebles, 1982b; Blumenthal *et al.*, 1984; Davis *et al.*, 1985). For these two reasons, at scales much smaller than the Hubble radius the equations of motion can be well approximated by Newtonian gravity.

In addition, all dark matter particle candidates are extremely light compared to the mass of typical astrophysical objects such as stars or galaxies, with an expected number density of a least 10^{50} particles per Mpc^3. Therefore, discreteness effects are negligible and collisionless dark matter can be well described in the fluid limit approximation.

In this section, we present the dynamics of gravitational instability in the framework of Newtonian gravity within a flat, expanding background and in the fluid limit approximation. It is of course possible to do a detailed relativistic treatment of structure formation dynamics and cosmological perturbation theory (Bardeen, 1980; Mukhanov, Feldman & Brandenberger, 1992; Malik & Wands, 2009) and to derive the Newtonian limit from general relativity (see e.g. Peebles, 1980).

1.3.1 The Vlasov-Poisson system

The cosmological Poisson equation. Let us consider a large number of particles that interact only gravitationally in an expanding universe. For a particle of velocity $\mathbf{v}$ at position $\mathbf{r}$, the action of all other particles can be treated as a smooth gravitational potential induced by the local mass density $\rho(\mathbf{r})$,

$$\phi(\mathbf{r}) = \mathrm{G} \int \frac{\rho(\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|} \, \mathrm{d}^3\mathbf{r}', \tag{1.59}$$

and the equation of motion reads

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\nabla_{\mathbf{r}}\phi = \mathrm{G} \int \frac{\rho(\mathbf{r}' - \mathbf{r})\,(\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} \, \mathrm{d}^3\mathbf{r}'. \tag{1.60}$$

Examining gravitational instabilities in the context of an expanding universe requires to consider the departure from the homogeneous Hubble flow. It is natural to describe the positions of particles in terms of their comoving coordinates $\mathbf{x}$ such that the physical coordinates are $\mathbf{r} = a\mathbf{x}$ and of the conformal time τ, defined by $\mathrm{d}t = a(\tau)\,\mathrm{d}\tau$. Hereafter, when there is no ambiguity, we will denote $\nabla \equiv \nabla_{\mathbf{x}}$ and $\Delta \equiv \Delta_{\mathbf{x}}$. The Jacobian of the spatial coordinate transformation is $|J| = a^3$ so that the right-hand side of the previous equation becomes

$$\mathrm{G} \int \frac{\rho(\mathbf{r}' - \mathbf{r})\,(\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} \, \mathrm{d}^3\mathbf{r}' = \mathrm{G} \int \frac{\rho(\mathbf{x}' - \mathbf{x})\, a\,(\mathbf{x}' - \mathbf{x})}{a^3|\mathbf{x}' - \mathbf{x}|^3} \, a^3 \, \mathrm{d}^3\mathbf{x}' \tag{1.61}$$

$$= \mathrm{G}a\bar{\rho} \int \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^3} \, \mathrm{d}^3\mathbf{x}' + \mathrm{G}a\bar{\rho} \int \delta(\mathbf{x}' - \mathbf{x}) \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^3} \, \mathrm{d}^3\mathbf{x}', \tag{1.62}$$

where we have introduced the density contrast $\delta(\mathbf{x})$, defined as

$$\rho(\mathbf{x}, t) \equiv \bar{\rho}(t) \left[1 + \delta(\mathbf{x}, t)\right], \tag{1.63}$$

where $\bar{\rho}(t) \propto 1/a^3$ (consequence of equation (1.3) with $w = 0$).

Velocities of particles are $\mathbf{v} = \dot{a}\mathbf{x} + a\,\mathrm{d}\mathbf{x}/\mathrm{d}t$, permitting us to define peculiar velocities as the difference between total velocities and the Hubble flow:

$$\mathbf{u} \equiv a\,\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v} - \dot{a}\mathbf{x}. \tag{1.64}$$

$\mathrm{d}\mathbf{v}/\mathrm{d}t$ is written in terms of $\mathbf{u}$ as

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \ddot{a}\mathbf{x} + \dot{a}\,\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \tag{1.65}$$

$$= \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \ddot{a}\mathbf{x} + \frac{\dot{a}}{a}\,\mathbf{u}. \tag{1.66}$$

By the use of the second Friedmann equation for the homogeneous background (equation (1.2)),

$$\ddot{a} = -\frac{4\pi\mathrm{G}}{3} a\bar{\rho}, \tag{1.67}$$

and Gauss's theorem,

$$\frac{4\pi}{3}\mathbf{x} = -\int \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^3} \mathrm{d}^3\mathbf{x}', \tag{1.68}$$

the term $\ddot{a}\mathbf{x}$ is equal to

$$\mathrm{G}a\bar{\rho} \int \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^3} \, \mathrm{d}^3\mathbf{x}' \equiv -\frac{1}{a}\nabla_{\mathbf{x}}\phi, \tag{1.69}$$

which leaves for the peculiar velocity the following equation of motion (see equations (1.60), (1.62) and (1.66)):

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \frac{\dot{a}}{a}\,\mathbf{u} = \mathrm{G}a\bar{\rho} \int \delta(\mathbf{x}' - \mathbf{x}) \frac{(\mathbf{x}' - \mathbf{x})}{|\mathbf{x}' - \mathbf{x}|^3} \, \mathrm{d}^3\mathbf{x}' \equiv -\frac{1}{a}\nabla_{\mathbf{x}}\Phi. \tag{1.70}$$

Here we have defined the cosmological gravitational potential Φ such that $\phi \equiv \phi + \Phi$ with, for the background,

$$\phi(\mathbf{x}) = \frac{4\pi\mathrm{G}}{3} a^2 \bar{\rho} \left(\frac{1}{2}|\mathbf{x}|^2\right) = -\mathcal{H}' \left(\frac{1}{2}|\mathbf{x}|^2\right), \quad \text{satisfying} \quad \Delta\phi = 4\pi\mathrm{G}a^2\bar{\rho}. \tag{1.71}$$

Using the overall Poisson equation, $\Delta_{\mathbf{r}}\phi = \Delta\phi/a^2 = 4\pi \mathrm{G}\bar{\rho}(1+\delta)$, we find that Φ follows a cosmological Poisson equation sourced only by density fluctuations, as expected:

$$\Delta\Phi = 4\pi \mathrm{G} a^2 \bar{\rho}\delta = \frac{3}{2}\Omega_{\mathrm{m}}(\tau)\mathcal{H}^2(\tau)\delta. \tag{1.72}$$

The second equality comes from the first Friedmann equation in a flat universe (equation (1.1) with $k = 0$).

The Vlasov equation. Looking at equation (1.70), the momentum of a single particle of mass m is identified as:

$$\mathbf{p} = ma\mathbf{u}, \tag{1.73}$$

and the equation of motion reads:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -m\nabla_{\mathbf{x}}\Phi = -ma\nabla_{\mathbf{r}}\Phi \quad \text{or} \quad \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = -ma\nabla_{\mathbf{x}}\Phi. \tag{1.74}$$

Let us now define the particle number density in phase space by $f(\mathbf{x}, \mathbf{p}, \tau)$. Phase-space conservation and Liouville's theorem imply the Vlasov equation (collisionless version of the Boltzmann equation):

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = \frac{\partial f}{\partial\tau} + \frac{\mathbf{p}}{ma}\cdot\nabla f - ma\nabla\Phi\cdot\frac{\partial f}{\partial\mathbf{p}} = 0. \tag{1.75}$$

Given equations (1.72) and (1.75), the Vlasov-Poisson system is closed.

1.3.2 Fluid dynamics approach, evolution equations in phase space

The Vlasov equation is very difficult to solve, since it is a partial differential equation involving seven variables, with non-linearity induced by the dependence of the potential Φ on the density through the Poisson equation. Its complicated structure prevents the analytic analysis of dark matter dynamics. In practice, we are usually not interested in solving the full phase-space dynamics, but only the evolution of the spatial distribution. It is therefore convenient to take momentum moments of the distribution function. This yields a fluid dynamics approach for the motion of collisionless dark matter. The zeroth-order momentum, by construction, relates the phase-space density to the density field,

$$\int f(\mathbf{x}, \mathbf{p}, \tau)\,\mathrm{d}^3\mathbf{p} \equiv \rho(\mathbf{x}, \tau). \tag{1.76}$$

The next order moments,

$$\int \frac{\mathbf{p}}{ma} f(\mathbf{x}, \mathbf{p}, \tau)\,\mathrm{d}^3\mathbf{p} \equiv \rho(\mathbf{x}, \tau)\mathbf{u}(\mathbf{x}, \tau), \tag{1.77}$$

$$\int \frac{\mathbf{p}_i\mathbf{p}_j}{m^2a^2} f(\mathbf{x}, \mathbf{p}, \tau)\,\mathrm{d}^3\mathbf{p} \equiv \rho(\mathbf{x}, \tau)\mathbf{u}_i(\mathbf{x}, \tau)\mathbf{u}_j(\mathbf{x}, \tau) + \sigma_{ij}(\mathbf{x}, \tau), \tag{1.78}$$

define the *peculiar velocity flow* $\mathbf{u}(\mathbf{x}, \tau)$ (average local velocity of particles in a region of space; for simplification, we use the same notation as the peculiar velocity of a single particle) and the *stress tensor* $\sigma_{ij}(\mathbf{x}, \tau)$ which can be related to the *velocity dispersion tensor*, $v_{ij}(\mathbf{x}, \tau)$, by $\sigma_{ij}(\mathbf{x}, \tau) \equiv \rho(\mathbf{x}, \tau)v_{ij}(\mathbf{x}, \tau)$.

By taking successive momentum moments of the Vlasov equation and integrating out phase-space information, a hierarchy of equations that couple successive moments of the distribution function can be constructed. The zeroth moment of the Vlasov equation gives the continuity equation,

$$\frac{\partial\delta(\mathbf{x}, \tau)}{\partial\tau} + \nabla\cdot\{[1 + \delta(\mathbf{x}, \tau)]\,\mathbf{u}(\mathbf{x}, \tau)\} = 0, \tag{1.79}$$

which describes the conservation of mass. Taking the first moment and subtracting $\bar{\rho}\,\mathbf{u}(\mathbf{x}, \tau)$ times the continuity equation yields the Euler equation,

$$\frac{\partial\mathbf{u}_i(\mathbf{x}, \tau)}{\partial\tau} + \mathcal{H}(\tau)\mathbf{u}_i(\mathbf{x}, \tau) + \mathbf{u}_j(\mathbf{x}, \tau)\cdot\nabla_j\mathbf{u}_i(\mathbf{x}, \tau) = -\nabla_i\Phi(\mathbf{x}, \tau) - \frac{1}{\rho(\mathbf{x}, \tau)}\nabla_j(\sigma_{ij}(\mathbf{x}, \tau)), \tag{1.80}$$

which describes the conservation of momentum. This equation is very similar to that of hydrodynamics, apart from an additional term which accounts for the expansion of the Universe and the fact that, contrary to perfect fluids, auto-gravitating systems may have an anisotropic stress tensor.

The infinite sequence of momentum moments of the Vlasov equation is usually truncated at this point and completed by fluid dynamics assumptions to close the system. Specifically, one postulates an Ansatz for the stress tensor, namely the equation of state of the cosmological fluid. For example, if the fluid is locally thermalized, the velocity dispersion becomes isotropic and proportional to the pressure (e.g. Bernardeau *et al.*, 2002):

$$\sigma_{ij} = -P\delta_{\mathrm{K}}^{ij}, \tag{1.81}$$

where δ_{K}^{ab} is a Kronecker symbol. Standard fluid dynamics also prescribes, with the addition of a viscous stress tensor, the following equation (e.g. Bernardeau *et al.*, 2002):

$$\sigma_{ij} = -P\delta_{\mathrm{K}}^{ij} + \zeta(\nabla\cdot\mathbf{u})\delta_{\mathrm{K}}^{ij} + \mu\left[\nabla_i\mathbf{u}_j + \nabla_j\mathbf{u}_i - \frac{2}{3}(\nabla\cdot\mathbf{u})\delta_{\mathrm{K}}^{ij}\right], \tag{1.82}$$

where ζ is the coefficient of bulk viscosity and μ is the coefficient of shear viscosity.

1.3.3 The single-stream approximation

At the early stages of cosmological gravitational instability, it is possible to further simplify and to rely on a different hypothesis, namely the *single-stream approximation*. At sufficiently large scales, gravity-induced cosmic flows will dominate over the velocity dispersion due to peculiar motions. The single-stream approximation consists in assuming that for CDM, velocity dispersion and pressure are negligible, i.e. $\sigma_{ij} = 0$, and that all particles have identical peculiar velocities. Hence, the density in phase space can be written

$$f(\mathbf{x},\mathbf{p},\tau) = \rho(\mathbf{x},\tau)\,\delta_{\mathrm{D}}[\mathbf{p} - ma\mathbf{u}(\mathbf{x})]\,. \tag{1.83}$$

Note, that from its definition, equation (1.78), the stress tensor characterizes the deviation of particle motions from a single coherent flow.

The single-stream approximation only works at the beginning of gravitational structure formation, when structures had no time to collapse and virialize. Because of non-linearity in the Vlasov-Poisson system, later stages will involve the superposition of three or more streams in phase space, indicating the break down of the approximation at increasingly larger scales. The breakdown of $\sigma_{ij} \approx 0$, describing the generation of velocity dispersion or anisotropic stress due to the multiple-stream regime, is generically known as *shell-crossing*. Beyond that point, the density in phase space exhibits no simple form, generally preventing further analytic analysis. This issue will be discussed further in chapters 2 and 6.

The single-stream approximation yields the following system of equations:

$$\frac{\partial\delta(\mathbf{x},\tau)}{\partial\tau} + \nabla\cdot\{[1+\delta(\mathbf{x},\tau)]\,\mathbf{u}(\mathbf{x},\tau)\} = 0, \tag{1.84}$$

$$\frac{\partial\mathbf{u}_i(\mathbf{x},\tau)}{\partial\tau} + \mathcal{H}(\tau)\mathbf{u}_i(\mathbf{x},\tau) + \mathbf{u}_j(\mathbf{x},\tau)\cdot\nabla_j\mathbf{u}_i(\mathbf{x},\tau) = -\nabla_i\Phi(\mathbf{x},\tau), \tag{1.85}$$

$$\Delta\Phi(\mathbf{x},\tau) = 4\pi\mathrm{G}a^2(\tau)\bar{\rho}(\tau)\delta(\mathbf{x},\tau). \tag{1.86}$$

It is a non-linear, closed system of equations involving the local density contrast, the local velocity field and the local gravitational potential.

There exists no general analytic solution to the fluid dynamics of collisionless self-gravitating CDM, even in the single-stream regime. However, literature provides several different analytic perturbative expansion techniques to yield approximate solutions for the dark matter dynamics, which we briefly review below (sections 1.4 and 1.5).

1.4 Eulerian perturbation theory

1.4.1 Eulerian linear perturbation theory

As mentioned above, at large scales and during the early stages of gravitational evolution, we expect the matter distribution to be smooth and to follow a single velocity stream. In this regime, it is therefore possible to linearize equation (1.84) and (1.85), assuming that fluctuations of density are small compared to the homogeneous contribution and that gradients of velocity fields are small compared to the Hubble parameter,

$$|\delta(\mathbf{x},\tau)| \ll 1, \tag{1.87}$$
$$|\nabla_j \mathbf{u}_i(\mathbf{x},\tau)| \ll \mathcal{H}(\tau). \tag{1.88}$$

We obtain the equation of motion in the *linear regime*,

$$\frac{\partial\delta(\mathbf{x},\tau)}{\partial\tau} + \theta(\mathbf{x},\tau) = 0, \tag{1.89}$$
$$\frac{\partial\mathbf{u}(\mathbf{x},\tau)}{\partial\tau} + \mathcal{H}(\tau)\mathbf{u}(\mathbf{x},\tau) = -\nabla\Phi(\mathbf{x},\tau), \tag{1.90}$$

where $\theta(\mathbf{x},\tau) \equiv \nabla\cdot\mathbf{u}(\mathbf{x},\tau)$ is the divergence of the velocity field. The velocity field, as any vector field, is completely described by its divergence $\theta(\mathbf{x},\tau)$ and its curl, referred to as the vorticity, $\mathbf{w}(\mathbf{x},\tau) \equiv \nabla\times\mathbf{u}(\mathbf{x},\tau)$, whose equations of motion follow from taking the divergence and the curl of equation (1.85) and using the Poisson equation:

$$\frac{\partial\theta(\mathbf{x},\tau)}{\partial\tau} + \mathcal{H}(\tau)\theta(\mathbf{x},\tau) + 4\pi \mathrm{G} a^2(\tau)\bar{\rho}(\tau)\delta(\mathbf{x},\tau) = 0, \tag{1.91}$$
$$\frac{\partial\mathbf{w}(\mathbf{x},\tau)}{\partial\tau} + \mathcal{H}(\tau)\mathbf{w}(\mathbf{x},\tau) = 0. \tag{1.92}$$

Since vorticity is not sourced in the linear regime, any initial vorticity rapidly decays due to the expansion of the Universe. Indeed, its evolution immediately follows from equation (1.92): $\mathbf{w}(\tau) \propto 1/a(\tau)$. In the non-linear regime, the emergence of anisotropic stress in the right-hand side of Euler's equation can lead to vorticity generation (Pichon & Bernardeau, 1999).

The density contrast evolution follows by replacing equation (1.89) and its time derivative in equation (1.91):

$$\frac{\partial^2\delta(\mathbf{x},\tau)}{\partial\tau^2} + \mathcal{H}(\tau)\frac{\partial\delta(\mathbf{x},\tau)}{\partial\tau} = 4\pi \mathrm{G} a^2(\tau)\bar{\rho}(\tau)\delta(\mathbf{x},\tau) = \frac{3}{2}\Omega_\mathrm{m}(\tau)\mathcal{H}^2(\tau)\delta(\mathbf{x},\tau). \tag{1.93}$$

1.4.2 The growth of fluctuations in linear theory

This linear equation allows us to look for different fluctuation modes, decoupling spatial and time contributions by writing $\delta(\mathbf{x},\tau) = D_1(\tau)\,\delta(\mathbf{x},0)$, where some "initial" reference time is labeled as 0 and where $D_1(\tau)$ is called the *linear growth factor*. The time dependence of the fluctuation growth rate satisfies

$$\frac{\mathrm{d}^2 D_1(\tau)}{\mathrm{d}\tau^2} + \mathcal{H}(\tau)\frac{\mathrm{d}D_1(\tau)}{\mathrm{d}\tau} = \frac{3}{2}\Omega_\mathrm{m}(\tau)\mathcal{H}^2(\tau)D_1(\tau), \tag{1.94}$$

regardless of $\mathbf{x}$ (or of the Fourier mode $\mathbf{k}$): in the linear regime, the growth of fluctuations is scale-independent. This equation, together with Friedmann's equations, equations (1.7) and (1.9), determines the growth of density perturbations in the linear regime as a function of cosmological parameters. There are two independent solutions, the fastest growing mode $D_1^{(+)}(\tau)$ and the slowest growing mode $D_1^{(-)}(\tau)$. The evolution of the density contrast is then given by:

$$\delta(\mathbf{x},\tau) = D_1^{(+)}(\tau)\delta_+(\mathbf{x}) + D_1^{(-)}(\tau)\delta_-(\mathbf{x}), \tag{1.95}$$

where $\delta_+(\mathbf{x})$ and $\delta_-(\mathbf{x})$ are two functions of position only describing the initial density field configuration.

In terms of the scale factor and using Friedmann's equations, equation (1.94) can be rewritten as

$$a^2\frac{\mathrm{d}^2 D_1}{\mathrm{d}a^2} + \left(\Omega_\Lambda(a) - \frac{\Omega_\mathrm{m}(a)}{2} + 2\right) a\frac{\mathrm{d}D_1}{\mathrm{d}a} = \frac{3}{2}\Omega_\mathrm{m}(a)D_1, \tag{1.96}$$

where the cosmological parameters $\Omega_\Lambda(a)$ and $\Omega_\mathrm{m}(a)$ now depend on the scale factor (for more details on this derivation and a generalization to time-varying dark energy models, see Percival, 2005b).

Using the linearized continuity equation, equation (1.89), the velocity divergence is given by

$$\theta(\mathbf{x},\tau) = -\mathcal{H}(\tau)\left[f(\Omega_i)\delta_+(\mathbf{x},\tau) + g(\Omega_i)\delta_-(\mathbf{x},\tau)\right]. \tag{1.97}$$

It does not depend directly on the linear growth factor of each mode, but on its logarithmic derivative, the exponent in the momentary power law relating D to a,

$$f(\Omega_i) \equiv \frac{1}{\mathcal{H}(\tau)}\frac{\mathrm{d}\ln D_1^{(+)}}{\mathrm{d}\tau} = \frac{\mathrm{d}\ln D_1^{(+)}}{\mathrm{d}\ln a}, \quad g(\Omega_i) \equiv \frac{1}{\mathcal{H}(\tau)}\frac{\mathrm{d}\ln D_1^{(-)}}{\mathrm{d}\tau} = \frac{\mathrm{d}\ln D_1^{(-)}}{\mathrm{d}\ln a}, \tag{1.98}$$

with $\delta_\pm(\mathbf{x},\tau) \equiv D_1^{(\pm)}(\tau)\delta_\pm(\mathbf{x})$.

We now review some cosmological models for which analytic expressions exist.

1. For a standard cold dark matter (SCDM) model, i.e. a particular case of an Einstein-de Sitter universe (Einstein & de Sitter, 1932) where dark matter is cold, the cosmological parameters are time-independent: $\Omega_\mathrm{m}(a) = 1$ and $\Omega_\Lambda(a) = 0$. Using equation (1.96), the evolution of the linear growth factor satisfies

$$a^2\frac{\mathrm{d}^2 D_1}{\mathrm{d}a^2} + \frac{3}{2}a\frac{\mathrm{d}D_1}{\mathrm{d}a} = \frac{3}{2}D_1. \tag{1.99}$$

Two independent solutions are

$$D_1^{(+)} \propto a, \quad f(\Omega_\mathrm{m}=1,\Omega_\Lambda=0)=1, \quad D_1^{(-)} \propto a^{-3/2}, \quad g(\Omega_\mathrm{m}=1,\Omega_\Lambda=0) = -\frac{3}{2}, \tag{1.100}$$

thus density fluctuations grow as the scale factor, $\delta \propto a$, once the decaying mode has vanished.

2. For an open cold dark matter (OCDM) model, the cosmological parameters are $\Omega_\mathrm{m}(a) < 1$ and $\Omega_\Lambda(a) = 0$. The solutions of equation (1.96) are (Groth & Peebles, 1975), with $x \equiv a(\tau)(1/\Omega_\mathrm{m}^{(0)} - 1)$,

$$D_1^{(+)} = 1 + \frac{3}{x} + 3\frac{(1+x)^{1/2}}{x^{3/2}}\ln\left[(1+x)^{1/2} - x^{1/2}\right], \quad D_1^{(-)} = \frac{(1+x)^{1/2}}{x^{3/2}}. \tag{1.101}$$

The dimensionless parameter g is calculated to be

$$g(\Omega_\mathrm{m},\Omega_\Lambda=0) = -\frac{\Omega_\mathrm{m}}{2} - 1, \tag{1.102}$$

and the dimensionless parameter f can be approximated by (Peebles, 1976, 1980)

$$f(\Omega_\mathrm{m},\Omega_\Lambda=0) \approx \Omega_\mathrm{m}^{3/5}. \tag{1.103}$$

As $\Omega_\mathrm{m} \to 0$ ($a \to \infty$ and $x \to \infty$), $D_1^{(+)} \to 1$ and $D_1^{(-)} \sim x^{-1}$: perturbations cease to grow.

3. For a universe with cold dark matter and a cosmological constant, $\Omega_\mathrm{m}(a) < 1$ and $0 < \Omega_\Lambda(a) \leq 1$ (ΛCDM model), allowing the possibility of a curvature term ($\Omega_\mathrm{tot}(a) = \Omega_\mathrm{m}(a) + \Omega_\Lambda(a) \neq 1$), the first Friedmann equation, equation (1.1), allows to write the Hubble parameter as

$$H(a) = H_0\sqrt{\Omega_\Lambda^{(0)} + (1-\Omega_\Lambda^{(0)} - \Omega_\mathrm{m}^{(0)})a^{-2} + \Omega_\mathrm{m}^{(0)}a^{-3}}. \tag{1.104}$$

It can be checked that this expression is a solution of equation (1.96). The decaying mode is then

$$D_1^{(-)} \propto H(a) = \frac{\mathcal{H}(a)}{a}. \tag{1.105}$$

Using this particular solution and the variation of parameters method, the other solution is found to be (Heath, 1977; Carroll, Press & Turner, 1992; Bernardeau *et al.*, 2002)

$$D_1^{(+)} \propto a^3 H^3(a)\int_0^a \frac{\mathrm{d}\tilde{a}}{\tilde{a}^3 H^3(\tilde{a})}. \tag{1.106}$$

Due to equations (1.7) and (1.105), one finds for arbitrary Ω_m and Ω_Λ,

$$g(\Omega_\mathrm{m},\Omega_\Lambda) = \Omega_\Lambda - \frac{\Omega_\mathrm{m}}{2} - 1, \tag{1.107}$$

and f can be approximated by (Lahav *et al.*, 1991)

$$f(\Omega_\mathrm{m}, \Omega_\Lambda) \approx \left[\frac{\Omega_\mathrm{m}^{(0)} a^{-3}}{\Omega_\mathrm{m}^{(0)} a^{-3} + (1 - \Omega_\Lambda^{(0)} - \Omega_\mathrm{m}^{(0)}) a^{-2} + \Omega_\Lambda^{(0)}} \right]^{3/5} \tag{1.108}$$

or (Lightman & Schechter, 1990; Carroll, Press & Turner, 1992)

$$f(\Omega_\mathrm{m}, \Omega_\Lambda) \approx \left[\frac{\Omega_\mathrm{m}^{(0)} a^{-3}}{\Omega_\mathrm{m}^{(0)} a^{-3} + (1 - \Omega_\Lambda^{(0)} - \Omega_\mathrm{m}^{(0)}) a^{-2} + \Omega_\Lambda^{(0)}} \right]^{4/7}. \tag{1.109}$$

For flat Universe, $\Omega_\mathrm{m} + \Omega_\Lambda = 1$, we have (Bouchet *et al.*, 1995; Bernardeau *et al.*, 2002)

$$f(\Omega_\mathrm{m}, \Omega_\Lambda = 1 - \Omega_\mathrm{m}) \approx \Omega_\mathrm{m}^{5/9}. \tag{1.110}$$

In the case of the Einstein-de Sitter universe, an interpretation of the growing and decaying modes can be easily given. Referring to the solution for the growth factor, equation (1.100), the initial density field (equation (1.95)) and the initial velocity field (equation (1.97)) are written

$$\delta_\mathrm{init}(\mathbf{x}) \equiv \delta(\mathbf{x}, 0) = \delta_+(\mathbf{x}) + \delta_-(\mathbf{x}), \tag{1.111}$$

$$\theta_\mathrm{init}(\mathbf{x}) \equiv \theta(\mathbf{x}, 0) = -\mathcal{H}(0) \left[\delta_+(\mathbf{x}) - \frac{3}{2} \delta_-(\mathbf{x}) \right], \tag{1.112}$$

if we assume that D_+ and D_- are normalized to unity at the initial time. These relations can be inverted to give

$$\delta_+(\mathbf{x}) = \frac{3}{5} \left(\delta_\mathrm{init}(\mathbf{x}) - \frac{2}{3} \frac{\theta_\mathrm{init}(\mathbf{x})}{\mathcal{H}(0)} \right), \tag{1.113}$$

$$\delta_-(\mathbf{x}) = \frac{2}{5} \left(\delta_\mathrm{init}(\mathbf{x}) + \frac{\theta_\mathrm{init}(\mathbf{x})}{\mathcal{H}(0)} \right). \tag{1.114}$$

From these expressions, the interpretation of the modes become clear. The sign is significant: recall that for a growing mode alone we would expect $\theta_\mathrm{init} = -\mathcal{H}(0)\delta_\mathrm{init}$ and for a decaying mode alone, $\theta_\mathrm{init} = 3/2\,\mathcal{H}(0)\delta_\mathrm{init}$. A pure growing mode corresponds to the case where the density and velocity fields are initially "in phase", in the sense that the velocity field converges towards the potential wells defined by the density field. A pure decaying mode corresponds to the case where the density and velocity fields are initially "opposite in phase", the velocity field being such that particles escape potential wells.

1.4.3 Eulerian perturbation theory at higher order

At higher order, Eulerian perturbation theory can be implemented by expanding the density and velocity fields,

$$\delta(\mathbf{x}, \tau) = \sum_{n=1}^{\infty} \delta^{(n)}(\mathbf{x}, \tau) = \delta^{(1)}(\mathbf{x}, \tau) + \delta^{(2)}(\mathbf{x}, \tau) + \dots, \tag{1.115}$$

$$\theta(\mathbf{x}, \tau) = \sum_{n=1}^{\infty} \theta^{(n)}(\mathbf{x}, \tau) = \theta^{(1)}(\mathbf{x}, \tau) + \theta^{(2)}(\mathbf{x}, \tau) + \dots, \tag{1.116}$$

where $\delta^{(1)}(\mathbf{x}, \tau)$ and $\theta^{(1)}(\mathbf{x}, \tau)$ are the linear order solution studied in the previous section. Focusing only on the growing mode, the first-order density field reads,

$$\delta^{(1)}(\mathbf{x}, \tau) = D_1(\tau) \delta_\mathrm{init}(\mathbf{x}), \tag{1.117}$$

with $D_1(\tau) \equiv D_1^{(+)}(\tau)$ and $\delta_\mathrm{init}(\mathbf{x}) = \delta_+(\mathbf{x})$. $\delta^{(2)}(\mathbf{x}, \tau)$ describes to leading order the non-local evolution of the density field due to gravitational interactions. It is found to be proportional to the *second-order growth factor*, $D_2(\tau)$, which satisfies the differential equation (equation 43 in Bouchet *et al.*, 1995)

$$a^2 \frac{\mathrm{d}^2 D_2}{\mathrm{d}a^2} + \left(\Omega_\Lambda(a) - \frac{\Omega_\mathrm{m}(a)}{2} + 2 \right) a \frac{\mathrm{d}D_2}{\mathrm{d}a} = \frac{3}{2} \Omega_\mathrm{m}(a) \left[D_2 - (D_1^{(+)})^2 \right]. \tag{1.118}$$

In the codes implemented for this thesis, we use the fitting function

$$D_2(\tau) \approx -\frac{3}{7} D_1^2(\tau)\Omega_\mathrm{m}^{-1/143}, \tag{1.119}$$

valid for a flat ΛCDM model (Bouchet *et al.*, 1995). Depending on the cosmological parameters, different expressions can be found in the literature (see e.g. Bernardeau *et al.*, 2002), but $D_2(\tau)$ always stays of the order of $D_1^2(\tau)$ as expected in perturbation theory.

A detailed presentation of non-linear Eulerian perturbation theory involves new types of objects (kernels, propagators, vertices) and is beyond the scope of this thesis. For an existing review, see e.g. Bernardeau *et al.* (2002).

1.5 Lagrangian perturbation theory

1.5.1 Lagrangian fluid approach for cold dark matter

As we have seen (section 1.3.2), our approach is based on the assumption that CDM is well described by a fluid. A way of looking at fluid motion is to focus on specific locations in space through which the fluid flows as time passes. It is then possible to study dynamics of density and velocity fields in this context, which constitutes the Eulerian point of view. We have developed Eulerian perturbation theory in section 1.4.

Alternatively, in fluid dynamics, one can choose to describe the field by following the trajectories of particles or fluid elements. This is the so-called Lagrangian description. The goal of this paragraph is to apply this description to the cosmological fluid and to build *Lagrangian perturbation theory* in this framework.

Mapping from Lagrangian to Eulerian coordinates. In Lagrangian description, the object of interest is not the position of particles but the *displacement field* $\Psi(\mathbf{q})$ which maps the initial comoving particle position $\mathbf{q}$ into its final comoving Eulerian position $\mathbf{x}$, (e.g. Buchert, 1989; Bouchet *et al.*, 1995; Bernardeau *et al.*, 2002):

$$\mathbf{x}(\mathbf{q},\tau) \equiv \mathbf{q} + \mathbf{\Psi}(\mathbf{q},\tau). \tag{1.120}$$

Let $J(\mathbf{q},\tau)$ be the Jacobian of the transformation between Lagrangian and Eulerian coordinates,

$$J(\mathbf{q},\tau) \equiv \left| \frac{\partial \mathbf{x}}{\partial \mathbf{q}} \right| = |\det \mathcal{D}| = |\det(\mathcal{I} + \mathcal{R})| \,, \tag{1.121}$$

where the deformation tensor $\mathcal{D}$ can be written as the identity tensor $\mathcal{I}$ plus the shear of the displacement,[3] $\mathcal{R} \equiv \partial\mathbf{\Psi}/\partial\mathbf{q}$. The Jacobian can be obtained by requiring that the Lagrangian mass element be conserved in the relationship between density contrast and trajectories:

$$\rho(\mathbf{x},\tau)\,\mathrm{d}^3\mathbf{x} = \rho(\mathbf{q})\,\mathrm{d}^3\mathbf{q} \quad \Rightarrow \quad \bar{\rho}(\tau)\,[1+\delta(\mathbf{x},\tau)]\,\mathrm{d}^3\mathbf{x} = \bar{\rho}(\tau)\mathrm{d}^3\mathbf{q}, \tag{1.122}$$

Hence,

$$J(\mathbf{q},\tau) = \frac{1}{1+\delta(\mathbf{x},\tau)} \quad \text{or} \quad \delta(\mathbf{x},\tau) = J^{-1}(\mathbf{q},\tau) - 1. \tag{1.123}$$

Note that this result (without the absolute value for J) is valid as long as no shell-crossing occurs. At the first crossing of trajectories, fluid elements with different initial positions $\mathbf{q}$ end up at the same Eulerian position $\mathbf{x}$ through the mapping in equation (1.120). The Jacobian vanishes and one expects a singularity, namely a collapse to infinite density. At this point, the description of dynamics in terms of a mapping does not hold anymore, the correct description involves a summation over all possible streams.

Equation of motion in Lagrangian coordinates. The equation of motion for a fluid element, equation (1.70), reads in conformal time,

$$\frac{\partial \mathbf{u}}{\partial \tau} + \mathcal{H}(\tau)\mathbf{u} = -\nabla_\mathbf{x}\Phi, \tag{1.124}$$

[3] $\mathcal{R}$ is mathematically a tensor. It is sometimes referred to as the tidal tensor and noted $\mathcal{T}$. We will avoid this nomenclature and notation here, so as not to introduce confusion with the Hessian of the gravitational potential $\mathcal{T} \equiv \partial^2\Phi/\partial\mathbf{x}^2$ (see section C.2).

where Φ is the cosmological gravitational potential and $\nabla_{\mathbf{x}}$ is the gradient operator in Eulerian comoving coordinates $\mathbf{x}$. Taking the divergence of this equation, noting that $\mathbf{u} = \mathrm{d}\mathbf{x}/\mathrm{d}\tau = \partial\mathbf{\Psi}/\partial\tau$, using equation (1.123) and the Poisson equation, equation (1.72), and multiplying by the Jacobian, we obtain

$$J(\mathbf{q},\tau)\,\nabla_{\mathbf{x}}\cdot\left[\frac{\partial^2\mathbf{\Psi}}{\partial\tau^2}+\mathcal{H}(\tau)\frac{\partial\mathbf{\Psi}}{\partial\tau}\right]=\frac{3}{2}\Omega_\mathrm{m}(\tau)\mathcal{H}^2(\tau)\left[J(\mathbf{q},\tau)-1\right]. \tag{1.125}$$

This equation shows the principal difficulty of the Lagrangian approach: the gradient operator has to be taken with reference to the Eulerian variable $\mathbf{x}$, which depends on $\mathbf{q}$ according to equation (1.120). Equation (1.125) can be rewritten in terms of Lagrangian coordinates only by using $(\nabla_{\mathbf{x}})_i = \left[\delta_\mathrm{K}^{ij}+\mathbf{\Psi}_{i,j}\right]^{-1}(\nabla_{\mathbf{q}})_j$, where $\mathbf{\Psi}_{i,j}\equiv\partial\mathbf{\Psi}_i/\partial\mathbf{q}_j=\mathcal{R}_{ij}$ are the shears of the displacement. The resulting non-linear differential equation for $\mathbf{\Psi}(\mathbf{q},\tau)$ is then solved perturbatively, expanding about its linear solution.

1.5.2 The Zel'dovich approximation

Displacement field in the Zel'dovich approximation. In Lagrangian approach, non-linearities of the dynamics are encoded in the relation between $\mathbf{q}$ and $\mathbf{x}$ (equation (1.120)) and in the relation between the displacement field and the local density (equation (1.123)). The Zel'dovich approximation (Zel'dovich, 1970; Shandarin & Zel'dovich, 1989, hereafter ZA) is first order Lagrangian perturbation theory. It consists of taking the linear solution of equation (1.125) for the displacement field while keeping the general equation with the Jacobian, equation (1.123), to reconstruct the density field. At linear order in the displacement field, the relation between the gradients in Eulerian and Lagrangian coordinates is $J(\mathbf{q},\tau)\nabla_{\mathbf{x}}\approx\nabla_{\mathbf{q}}$, and the first-order Jacobian is $J(\mathbf{q},\tau)\approx 1+\nabla_{\mathbf{q}}\cdot\mathbf{\Psi}$. The equation to solve becomes

$$\nabla_{\mathbf{q}}\cdot\left[\frac{\partial^2\mathbf{\Psi}}{\partial\tau^2}+\mathcal{H}(\tau)\frac{\partial\mathbf{\Psi}}{\partial\tau}\right]=\frac{3}{2}\Omega_\mathrm{m}(\tau)\mathcal{H}^2(\tau)\left(\nabla_{\mathbf{q}}\cdot\mathbf{\Psi}\right). \tag{1.126}$$

The addition of any divergence-free displacement field to a solution of the previous equation will also be a solution. In the following, we remove this indeterminacy by assuming that the movement is potential, i.e. $\nabla_{\mathbf{q}}\times\mathbf{\Psi}=0$. Introducing the divergence of the Lagrangian displacement field, $\psi\equiv\nabla_{\mathbf{q}}\cdot\mathbf{\Psi}$, one has to solve,

$$\psi''+\mathcal{H}(\tau)\psi'=\frac{3}{2}\Omega_\mathrm{m}(\tau)\mathcal{H}^2(\tau)\psi. \tag{1.127}$$

Therefore, the linear solution of equation (1.125) is separable into a product of a temporal and a spatial contribution. It can be written as $\mathbf{\Psi}^{(1)}(\mathbf{q},\tau)$ such that

$$\psi^{(1)}(\mathbf{q},\tau)\equiv\nabla_{\mathbf{q}}\cdot\mathbf{\Psi}^{(1)}(\mathbf{q},\tau)=-D_1(\tau)\,\delta(\mathbf{q}), \tag{1.128}$$

where $D_1(\tau)$ denotes the linear growth factor studied in section 1.4.1 and $\delta(\mathbf{q})$ describes the growing mode of the initial density contrast field in Lagrangian coordinates. This can be checked in equation (1.127) using the differential equation satisfied by the growth factor, equation (1.94). The above choice for the spatial contribution permits to recover the linear Eulerian behaviour, since initially $\delta(\mathbf{x})\approx D_1(\tau)\delta(\mathbf{q})\approx(1+\psi)^{-1}-1\approx-\psi$.

Note that the evolution of fluid elements at linear order is *local evolution*, i.e. it does not depend on the behavior of the rest of fluid elements. We have assumed that at linear order, the displacement field is entirely determined by its divergence, i.e. that vorticity vanishes. As we have already noted from equation (1.92), in the linear regime, any initial vorticity decays away due to the expansion of the Universe. Thus, one might consider that the solutions will apply anyway, even if vorticity is initially present, because at later times it will have negligible effect. Similarly, we have neglected the effect of the decaying mode in equation (1.95).

Shell-crossing in the Zel'dovich approximation. Since the displacement field in the ZA is curl-free, it is convenient to introduce the potential from which it derives, $\phi^{(1)}(\mathbf{q})$, such that $\mathbf{\Psi}^{(1)}(\mathbf{q},\tau)=-D_1(\tau)\nabla_{\mathbf{q}}\phi^{(1)}(\mathbf{q})$. At linear order in the displacement field, its shear $\mathcal{R}\equiv\partial\mathbf{\Psi}^{(1)}/\partial\mathbf{q}$ is equal to $-D_1(\tau)\mathrm{H}(\phi^{(1)}(\mathbf{q}))$. Let $\lambda_1(\mathbf{q})\leq\lambda_2(\mathbf{q})\leq\lambda_3(\mathbf{q})$ be the local eigenvalues of the Hessian of the Zel'dovich potential $\phi^{(1)}(\mathbf{q})$. At conformal time τ, these values have grown of a factor $-D_1(\tau)$ to give the eigenvalues of the shear of the displacement $\mathcal{R}$. Using equation (1.123), the density contrast may then be written as (e.g. Bouchet *et al.*, 1995; Bernardeau *et al.*, 2002)

$$1+\delta(\mathbf{x},\tau)=\frac{1}{\left[1-\lambda_1(\mathbf{q})D_1(\tau)\right]\left[1-\lambda_2(\mathbf{q})D_1(\tau)\right]\left[1-\lambda_3(\mathbf{q})D_1(\tau)\right]}. \tag{1.129}$$

This equation allows an interpretation of what happens at shell-crossing in the ZA. If all eigenvalues λ_i are negative, this is a developing underdense region, eventually reaching $\delta = -1$. If λ_3 only is positive, when $\lambda_3 D_1(\tau) \rightarrow 1$, the ZA leads to a planar collapse to infinite density along the axis of λ_3 and the formation of a two-dimensional "cosmic pancake". In the case when two eigenvalues are positive, $\lambda_2, \lambda_3 > 0$, there is collapse to a filament. The case $\lambda_1, \lambda_2, \lambda_3 > 0$ leads to gravitational collapse along all directions (spherical collapse if $\lambda_1 \approx \lambda_2 \approx \lambda_3$). This picture of gravitational structure formation leads to a cosmic web classification algorithm, which labels different regions either as voids, sheets, filaments, or halos (see Hahn *et al.*, 2007a; Lavaux & Wandelt, 2010, and section C.2).

1.5.3 Second-order Lagrangian perturbation theory

Displacement field in second-order Lagrangian perturbation theory. The Zel'dovich approximation being local, it fails at sufficiently non-linear stages when particles are forming gravitationally bound structures instead of following straight lines. Already second-order Lagrangian perturbation theory (hereafter 2LPT) provides a remarkable improvement over the ZA in describing the global properties of density and velocity fields (Melott, Buchert & Weiß, 1995). The solution of equation (1.125) up to second order takes into account the fact that gravitational instability is *non-local*, i.e. it includes the correction to the ZA displacement due to gravitational tidal effects. It reads

$$\mathbf{x}(\tau) = \mathbf{q} + \mathbf{\Psi}(\mathbf{q},\tau) = \mathbf{q} + \mathbf{\Psi}^{(1)}(\mathbf{q},\tau) + \mathbf{\Psi}^{(2)}(\mathbf{q},\tau), \quad \text{or} \quad \mathbf{\Psi}(\mathbf{q},\tau) = \mathbf{\Psi}^{(1)}(\mathbf{q},\tau) + \mathbf{\Psi}^{(2)}(\mathbf{q},\tau), \tag{1.130}$$

where the divergence of the first order solution is the same as in the ZA (equation (1.128)),

$$\psi^{(1)}(\mathbf{q},\tau) = \nabla_{\mathbf{q}} \cdot \mathbf{\Psi}^{(1)}(\mathbf{q},\tau) = -D_1(\tau)\,\delta(\mathbf{q}), \tag{1.131}$$

and the divergence of the second order solution describes the tidal effects,

$$\psi^{(2)}(\mathbf{q},\tau) = \nabla_{\mathbf{q}} \cdot \mathbf{\Psi}^{(2)}(\mathbf{q},\tau) = \frac{1}{2}\frac{D_2(\tau)}{D_1^2(\tau)} \sum_{i\neq j} \left[\mathbf{\Psi}^{(1)}_{i,i}\mathbf{\Psi}^{(1)}_{j,j} - \mathbf{\Psi}^{(1)}_{i,j}\mathbf{\Psi}^{(1)}_{j,i} \right], \tag{1.132}$$

where $\mathbf{\Psi}^{(1)}_{k,l} \equiv \partial \mathbf{\Psi}^{(1)}_k / \partial \mathbf{q}_l$ and $D_2(\tau)$ denotes the second-order growth factor, defined in section 1.4.3.

Lagrangian potentials. Since Lagrangian solutions up to second order are irrotational (Melott, Buchert & Weiß, 1995; Buchert, Melott & Weiß, 1994; Bernardeau *et al.*, 2002; this is assuming that initial conditions are only in the growing mode, in the same spirit as neglecting completely the decaying vorticity), it is convenient to define the Lagrangian potentials $\phi^{(1)}$ and $\phi^{(2)}$ from which $\mathbf{\Psi}^{(1)}$ and $\mathbf{\Psi}^{(2)}$ derive, so that in 2LPT,

$$\mathbf{\Psi}^{(1)}(\mathbf{q},\tau) = -D_1(\tau)\nabla_{\mathbf{q}}\phi^{(1)}(\mathbf{q}) \quad \text{and} \quad \mathbf{\Psi}^{(2)}(\mathbf{q},\tau) = D_2(\tau)\nabla_{\mathbf{q}}\phi^{(2)}(\mathbf{q}). \tag{1.133}$$

Since $\mathbf{\Psi}^{(1)}$ is of order $D_1(\tau)$ (equation (1.131)) and $\mathbf{\Psi}^{(2)}(\tau)$ is of order $D_2(\tau)$ (equation (1.132)), the above potentials are time-independent. They satisfy Poisson-like equations (Buchert, Melott & Weiß, 1994),

$$\Delta_{\mathbf{q}}\phi^{(1)}(\mathbf{q}) = \delta(\mathbf{q}), \tag{1.134}$$

$$\Delta_{\mathbf{q}}\phi^{(2)}(\mathbf{q}) = \sum_{i>j} \left[\phi^{(1)}_{,ii}(\mathbf{q})\phi^{(1)}_{,jj}(\mathbf{q}) - (\phi^{(1)}_{,ij}(\mathbf{q}))^2 \right]. \tag{1.135}$$

The mapping from Eulerian to Lagrangian, equation (1.130), thus reads

$$\mathbf{x}(\tau) = \mathbf{q} - D_1(\tau)\nabla_{\mathbf{q}}\phi^{(1)}(\mathbf{q}) + D_2(\tau)\nabla_{\mathbf{q}}\phi^{(2)}(\mathbf{q}). \tag{1.136}$$

Velocity field in second-order Lagrangian perturbation theory. Taking the derivative of the previous equation yields for the velocity field,

$$\mathbf{u} = -f_1(\tau)D_1(\tau)\mathcal{H}(\tau)\nabla_{\mathbf{q}}\phi^{(1)}(\mathbf{q}) + f_2(\tau)D_2(\tau)\mathcal{H}(\tau)\nabla_{\mathbf{q}}\phi^{(2)}(\mathbf{q}). \tag{1.137}$$

which involves the logarithmic derivatives of the growth factors, $f_i \equiv \mathrm{d}\ln D_i/\mathrm{d}\ln a$, well approximated in a flat ΛCDM model by (Bouchet *et al.*, 1995)

$$f_1 \approx \Omega_\mathrm{m}^{5/9} \quad \text{and} \quad f_2 \approx 2\,\Omega_\mathrm{m}^{6/11} \approx 2\,f_1^{54/55}. \tag{1.138}$$

Other expressions for different cosmologies can be found in Bouchet *et al.* (1995); Bernardeau *et al.* (2002).

1.6 Non-linear approximations to gravitational instability

When fluctuations become strongly non-linear in the density field, Eulerian perturbation theory breaks down. Lagrangian perturbation theory is often more successful, since the Lagrangian picture is intrinsically non-linear in the density field (see e.g. equation (1.125)). A small perturbation in the Lagrangian displacement field carries a considerable amount of non-linear information about the corresponding Eulerian density and velocity fields. However, at some point, computers are required to study gravitational instability (in particular through N-body simulations), the important drawback being that the treatment becomes numerical instead of analytical. We will adopt this approach in this thesis. However, several non-linear approximations to the equations of motion have been suggested in the literature to allow the extrapolation of analytical calculations in the non-linear regime. We now briefly review some of them (see also Melott, 1994; Sahni & Coles, 1995).

Non-linear approximations consist of replacing one of the equations of the dynamics (Poisson – equation (1.72) –, continuity – equation (1.79) – or Euler – equation (1.80)) by a different assumption.[4] In general, the Poisson equation is replaced (Munshi & Starobinsky, 1994). These modified dynamics are often local, in the sense described above for the ZA, in order to provide a simpler way of calculating the evolution of fluctuations than the full non-local dynamics.

1.6.1 The Zel'dovich approximation as a non-linear approximation

As we have seen in section 1.3, in Eulerian dynamics, non-linearity is encoded in the Poisson equation, equation (1.72), $\Delta\Phi = 4\pi \mathrm{G} a^2 \bar{\rho}\delta$. The goal of this paragraph is to see what replaces the Poisson equation in the Eulerian description of the ZA. From this point of view, the ZA is the original non-linear Eulerian approximation, and it remains one of the most famous.

If we restrict our attention to potential movements, the peculiar velocity field $\mathbf{u}$ is irrotational. It can be written as the gradient of a velocity potential,

$$\mathbf{u} = -\frac{\nabla_{\mathbf{x}} V}{a}. \tag{1.139}$$

As discussed before, the main reason to restrict to this case is the decay of vortical perturbations.

It is then possible to postulate various forms for the velocity potential V. The ZA corresponds to the Ansatz (Munshi & Starobinsky, 1994; Hui & Bertschinger, 1996; appendix B in Scoccimarro, 1997)

$$V = \frac{2fa}{3\Omega_\mathrm{m}\mathcal{H}}\Phi, \tag{1.140}$$

where Φ is the cosmological gravitational potential and f is the logarithmic derivative of the linear growth factor. The Zel'dovich approximation is therefore equivalent to the replacement of the Poisson equation by

$$\mathbf{u} = -\frac{2f}{3\Omega_\mathrm{m}\mathcal{H}}\nabla\Phi. \tag{1.141}$$

This can be explicitly checked as follows. Combining equations (1.124) and (1.141), one gets

$$\frac{\partial \mathbf{u}}{\partial \tau} + \mathcal{H}\mathbf{u} = \frac{3\Omega_\mathrm{m}\mathcal{H}}{2f}\mathbf{u}. \tag{1.142}$$

Then, noting that $\nabla_{\mathbf{q}} \cdot \mathbf{u} = \psi'$, the differential equation for ψ is

$$\psi'' + \mathcal{H}\psi' = \frac{3\Omega_\mathrm{m}\mathcal{H}}{2f}\psi', \tag{1.143}$$

Using the time evolution of D_1 (equation (1.94)) and the identity $D_1' = \mathcal{H} f D_1$, one can check that the Zel'dovich solution, $\psi = -D_1\delta(\mathbf{q})$ indeed verifies the above equation.

Equation (1.141) means that at linear order, particles just go straight (in comoving coordinates) in the direction set by their initial velocity. In the Zel'dovich approximation, the proportionality between velocity field and gravitational field always holds (not just to first order in $\mathbf{\Psi}$).

[4] In this section, we have come back to a Eulerian description of the cosmological fluid.

Note that during the matter era, $a \propto t^{2/3}$ and thus $\mathcal{H} \equiv \dot{a} = 2a/(3t)$, which means that an equivalent form for the ZA Ansatz is

$$V = \frac{f}{\Omega_\mathrm{m}} \Phi t \approx \Phi t. \tag{1.144}$$

The ZA is a local approximation that represents exactly the true dynamics in one-dimensional collapse (Buchert, 1989; Yoshisato *et al.*, 2006). It is also possible to formulate local approximations that besides describing correctly planar collapse like the ZA, are suited for cylindrical or spherical collapse (leading to the formation of cosmic filaments and halos, in addition to cosmic pancakes). These approximations, namely the "non-magnetic" approximation (NMA, Bertschinger & Jain, 1994) and the "local tidal" approximation (LTA, Hui & Bertschinger, 1996), are not straightforward to implement for the calculation of statistical properties of density and velocity fields.

1.6.2 Other velocity potential approximations

Some other possibilities for the velocity potential can be found in literature (Coles, Melott & Shandarin, 1993; Munshi & Starobinsky, 1994). The frozen flow (FF) approximation (Matarrese *et al.*, 1992) postulates

$$V = \Phi^{(1)} t, \tag{1.145}$$

where $\Phi^{(1)}$ is the first-order solution (the linear approximation) for the gravitational potential. It satisfies the Poisson equation in the linear regime,

$$\Delta\Phi^{(1)} = \frac{3}{2}\Omega_\mathrm{m}(\tau)\mathcal{H}^2(\tau)\delta_1(\mathbf{x},\tau), \tag{1.146}$$

where $\delta_1(\mathbf{x},\tau) = D_1(\tau)\,\delta_1(\mathbf{x})$ is the linearly extrapolated density field. In FF, the Poisson equation is replaced by the analog of equation (1.141), substituting equation (1.145),

$$\mathbf{u} = -\frac{2f}{3\Omega_\mathrm{m}\mathcal{H}}\nabla\Phi^{(1)}, \tag{1.147}$$

or, by taking the divergence and using equation (1.146),

$$\theta(\mathbf{x},\tau) = -\mathcal{H}(\tau) f\,\delta_1(\mathbf{x},\tau). \tag{1.148}$$

The physical meaning of this approximation is that the velocity field is assumed to remain linear while the density field is allowed to explore the non-linear regime.

In the linearly-evolving potential (LEP) approximation (Brainerd, Scherrer & Villumsen, 1993; Bagla & Padmanabhan, 1994), the gravitational potential is instead assumed to remain the same as in the linear regime; therefore, the Poisson equation is replaced by

$$\Phi = \Phi^{(1)}, \quad \Delta\Phi = \frac{3}{2}\Omega_\mathrm{m}(\tau)\mathcal{H}^2(\tau)\delta_1(\mathbf{x},\tau). \tag{1.149}$$

The idea is that since $\Phi \propto \delta/k^2$ in Fourier space, the gravitational potential is dominated by the long-wavelength modes more than the density field, and therefore it ought to obey linear perturbation theory to a better approximation.

1.6.3 The adhesion approximation

All the above approximations (ZA, NMA, LTA, FF, LP) are local, which means that we neglect the self-gravity of inhomogeneities. A significant problem of the ZA, and of subsequent variations, is the fact that after shell-crossing, matter continues to flow throughout the newly-formed structure, which should instead be gravitationally bound. This phenomenon washes out cosmic structures on small scales.

A possible phenomenological solution is to add a viscosity term to the single-stream Euler equation, equation (1.85), which then becomes Burgers' equation,

$$\frac{\partial \mathbf{u}_i(\mathbf{x},\tau)}{\partial\tau} + \mathcal{H}(\tau)\mathbf{u}_i(\mathbf{x},\tau) + \mathbf{u}_j(\mathbf{x},\tau)\cdot\nabla_j\mathbf{u}_i(\mathbf{x},\tau) = -\nabla_i\Phi(\mathbf{x},\tau) + \upsilon\,\Delta\mathbf{u}_i(\mathbf{x},\tau). \tag{1.150}$$

This is the so-called *adhesion approximation* (Kofman & Shandarin, 1988; Gurbatov, Saichev & Shandarin, 1989; Kofman *et al.*, 1992; Valageas & Bernardeau, 2011; Hidding *et al.*, 2012). For a potential flow, it can be reduced to a linear diffusion equation, and therefore solved exactly. Surprisingly, in the adhesion approximation, the dynamical equations describing the evolution of the self-gravitating cosmological fluid can be written in the form of a Schrödinger equation coupled to a Poisson equation describing Newtonian gravity (Short & Coles, 2006b). The dynamics can therefore be studied with the tools of wave mechanics. An alternative to the adhesion model is the free-particle approximation (FPA), in which the artificial viscosity term in Burgers' equation is replaced by a non-linear term known as the *quantum pressure*. This also leads to a free-particle Schrödinger equation (Short & Coles, 2006b,a).

Comparisons of the adhesion approximation to full-gravitational numerical simulations show an improvement over the ZA at small scales, even if the fragmentation of structures into dense clumps is still underestimated (Weinberg & Gunn, 1990). At weakly non-linear scales, the adhesion approximation is essentially equal to the ZA.